

Background on the curve

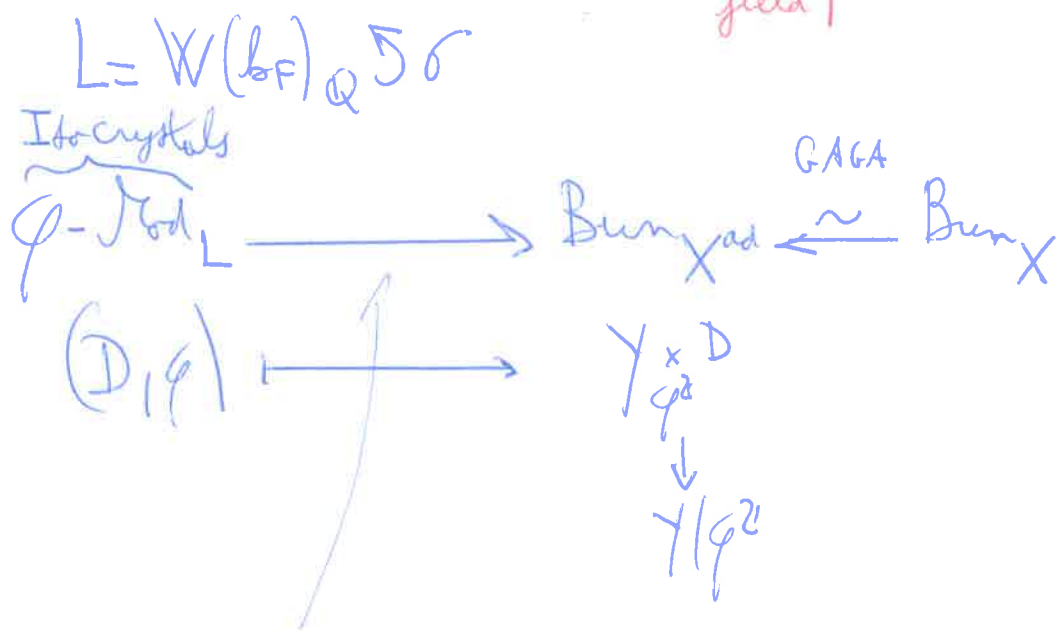
F perfectoid char. p. Joint work w/ Fontaine & curve X_F

[3 incarnations]

Adic perfectoid curve	Schematic curve	Diamond curve
$X_F^{\text{ad}} = Y_F / \varphi^{\mathbb{Z}}$ $Y_F = \text{Spa}(W(\mathbb{O}_F)) \setminus V(\varphi[\mathbb{O}_F])$ = punctured open disk variable φ Coeff. in F	$X_F = \text{Proj} \left(\bigoplus_{d \geq 0} H^0(X^{\text{ad}}, \mathcal{O}(d)) \right)$ $\mathcal{O}(d) = Y \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1$ (line bundle) noetherian regular scheme of dim. 1	$X^{\diamond} = (\text{Spa}(F) \times \text{Spa}(\mathbb{Q})^{\diamond}) / \varphi^{\mathbb{Z}}$ $\varphi = \text{Fib}_F \times \text{Id} = \text{Id} \times \text{Fib}_{\mathbb{Q}}^{-1}$

Falg. closed $b_F \hookrightarrow \mathbb{O}_F$

(Choose a section of the projection to the residue field)



Th (Fontaine-F.). This is essentially surjective.

Classification of G -torsors

G reductive gp. / \mathbb{Q}_F

$$G\text{-bundles}/X = \begin{cases} G\text{-torsors on } X & (\text{locally trivial for the étale top.}) \\ \text{"} \\ \text{Bundles w.t. a } G\text{-structure (Tannakian sense)} \end{cases}$$

$\text{Hom}(G, L) \rightsquigarrow E_G$ G -torsor on X_F

$$\begin{array}{ccc} \text{Rep}_{\mathbb{Q}_F} G & \longrightarrow & \mathcal{G}\text{-Mod}_L & \xrightarrow{\text{preceding functor}} & \text{Bun } X \\ (V, \rho) & \longmapsto & (V_L, \rho(\cdot|_G)) & & \end{array}$$

Th: * Any G -torsor on X is of this form Kottwitz set
 * canonical bijection $H^1(X, G) \xrightarrow{\sim} B(G) = G(L)/G\text{-conjugacy}$

does not depend on $\text{hom } G \rightarrow G_F$

Nice features: * $H^1(\mathbb{Q}_F, G) \xrightarrow{\sim} B(G)$

$\{[g] \in B(G) \mid \gamma_g = 1\} =$ unit root G -isocrystals
 \uparrow isomorphism (gives Dieudonné-Manin slope de composition)

given by $H^1(\mathbb{Q}_F, G) \subset H^1(X, G)$
 \uparrow pullback of G -torsors

* G quasi-split. \hookrightarrow basic $\iff E_G$ semi-stable in Atiyah-Bott sense
 there's a dictionary between Kottwitz and Atiyah-Bott reduction

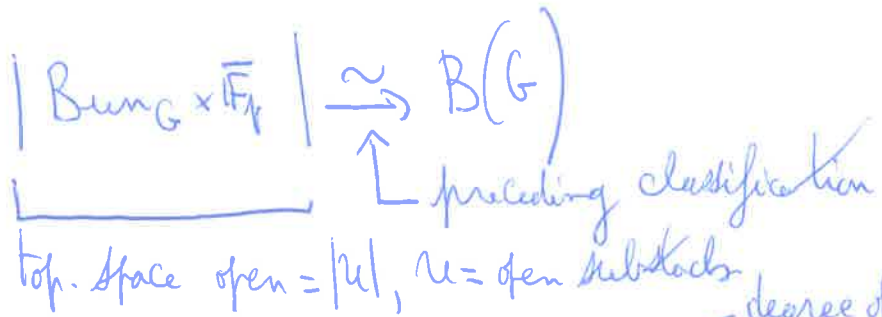
Bun_G

Perf = \mathbb{F}_p -perfectoid spaces + pro-étale topology (or faithful)

$\mathbb{S} \ni S \rightsquigarrow X_S^{\text{ad}} = \text{adic space}/\mathbb{Q}_p$
 = family of curves of the preceding type parametrized by S

Th: $S \mapsto \{ \text{G-bundles on } X_S \}$ is a stack (Scholze: even for the faithful top.)
 $\text{Rep}_{\mathbb{Q}_p} G \xrightarrow{\otimes\text{-exact}} \text{v. b. on } X_S$

Let's try to understand this stack



Connected Components: $\chi: B(G) \rightarrow \pi_1(G)_{\Gamma} = X^*(Z(\widehat{G})^{\Gamma})$
 ↖ degree of a v.b. if $G = GL_n$
 ↑ can be interpreted as a generalized first Chern class of a v.b.

Th (in progress). This is locally constant on $| \text{Bun}_G \times \overline{\mathbb{F}_p} |$

Conjecture: Fibers of $K =$ Connected Components

(Should not be difficult)

H.N. Stratification:

$$|Bun_G \times \overline{\mathbb{F}_t}| \longrightarrow \text{Hom}(\mathbb{D}_{\overline{\mathbb{Q}_t}}, G_{\overline{\mathbb{Q}_t}}) / G_{\overline{\mathbb{Q}_t}}\text{-conj.}$$

$$B(G) \ni [b] \longmapsto [\gamma_b]$$

Kedlaya-Liu (GL_n): this is semi-continuous.

S.S. locus = open strata.

S.S. locus: $[b] \in B(G) \rightsquigarrow \mathcal{E}_b = G\text{-torsion } X_{\mathbb{Q}_t^b}$

$\widetilde{\mathcal{J}}_b := \text{Aut}(\mathcal{E}_b) = \text{group sheaf on Perf}_{\mathbb{Q}_t^b}$
 " diamond

$$\pi_0(\widetilde{\mathcal{J}}_b) = \mathcal{J}_b(\mathbb{Q}_t) = G\text{-Centralizer of } b \text{ in } G(L)$$

$\widetilde{\mathcal{J}}_b^{\circ} = \text{unipotent}$ (group-diamond)

$$\text{Cen. Aut}(G_X \oplus G_X(\pm)) = \begin{pmatrix} \mathbb{Q}_t^{\times} & (B^+_{\text{basic}})^{\varphi=1} \\ 0 & \mathbb{Q}_t^{\times} \end{pmatrix} \leftarrow \text{cryptic remark}$$

$\mathcal{D}(\mathbb{Q}_t)$ (loc. analytic distributions)

$$b \text{ basic} \Rightarrow \widetilde{\mathcal{J}}_b^{\circ} = \{1\}$$

$$\text{Aut}(\mathcal{E}_b) = \widetilde{\mathcal{J}}_b(\mathbb{Q}_t)$$

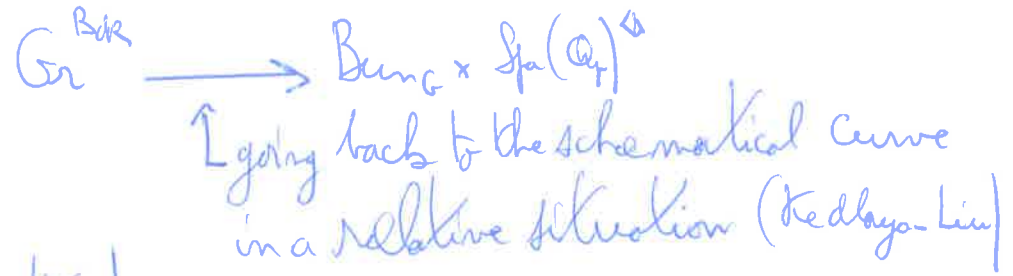
inner form of G .

any loc. profinite top-group defines a group sheaf on the pro-finite site

Using Kedlaya-Liu: Th. $[b]$ basic \Rightarrow $Bun_{G/\mathbb{F}_q}^{[G]}$
 \uparrow $G \text{ m, } b=1$ \parallel
 $B(J_0(\mathbb{Q}_p))$

S.S. stack = zero dimensional
dim. stratum associated to $\mathcal{O}_X \oplus \mathcal{O}_X(1) = -1$
deeper in H.N. stack \Rightarrow the dimension goes to $-\infty$.

Uniformization a la Beauville-Laszotz:



Th. F alg. closed. $\infty \in |X_F|$
schematical curve
 $E = G$ -torsor on X_F . Then $E|_{X_F \rightarrow \infty}$ is trivial.

analog of Drinfeld-Simpson

Should imply, using $G_{\mathbb{R}}^{Bde}$ is a diamond, Scholze.
 $Bun_G =$ perfectoid stack
(hug a pro-etale cover that is a presentation by a perfectoid space)

The Conjecture

Motivation: * F perfectoid field

$$\text{Pic}^0(X_F) = \text{Hom}(\pi_1^{\text{gdo}}(X_F), \mathbb{Q}_F^\times)$$

→ Starting point of geometric class field theory of $X =$ "classical curve" → pure inner forms

* Kottwitz Conjecture (generalizing Vogan's Conjecture): local

Langlands Correspondence for extended pure inner forms of G

J_b with b basic

→ Kaeltha works (depth 0 case)

Rem: b basic

$J_b =$ inner form of G (not pure inner form)

$$J_b \times X \cong \text{pure inner form of } G \times X$$

twisted by $[\epsilon_b] \in H^1(X, G)$

$$* \underbrace{\mathbb{Q}_F^\diamond / \varphi^2 = X_{\mathbb{F}_1}}_{\pi_1 = \text{Gal}(\overline{\mathbb{Q}_F} / \mathbb{Q}_F)} \rightsquigarrow "X_S = S \times \mathbb{Q}_F^\diamond / \varphi^2"$$

* Frenkel - Gaitsgory - Vilonen ...

$$\hat{G} = \overline{\mathbb{Q}_F}\text{-Langlands dual} \rightsquigarrow \text{group in } (\mathbb{Q}_F^\diamond / \varphi^2)_{\text{ét}}$$

$$\begin{matrix} \uparrow \\ \hat{G} \\ \uparrow \\ \Gamma = \text{Gal}(\overline{\mathbb{Q}_F} / \mathbb{Q}_F) \end{matrix}$$

Canonically defined via geometric Satake (Scholze via G^{Bar})

Conjecture (rough form):

Groupoid of \widehat{G} -torsors on $\mathbb{A}_f^1/p^2 \longrightarrow$ Perverse sheaves on $\text{Bun}_{\mathbb{A}_f^1/p^2}$ -Hecke equivariants

Conjecture: G quasi-split. $(B, \psi) =$ Whittaker datum

$\mathcal{L}_G = [\text{Hom}(W_{\text{loc}}, {}^L G) / \widehat{G}\text{-conj.}] =$ groupoid of Langlands parameters

$\mathcal{L}_G^{\text{disc}} =$ discrete Langlands parameters
 $\varphi: W_{\text{loc}} \rightarrow {}^L G$ $S_\varphi = \text{Cent}_G(\varphi)$
 \cup
 $Z(\widehat{G})^\Gamma$ $S_\varphi / Z(\widehat{G})^\Gamma$ finite

There exist a functor



such that:

1) If $\xi \in X^*(Z(\widehat{G})^\Gamma)$ and $\text{Bun}_{G/\overline{\mathbb{F}}_q}^\xi = X^{-1}(\xi)$ (associated connected component)

$J_\varphi^* \upharpoonright_{\text{Bun}^\xi}$
 \uparrow
 $S_\varphi \supset Z(\widehat{G})^\Gamma$ acts via ξ

2) $\mathcal{I}_\varphi / \text{Bun}^{ss}$ is a local system and if $j: \text{Bun}^{ss, \text{open}} \hookrightarrow \text{Bun}^{ss}$

$$\mathcal{I}_\varphi = j_! * j^* \mathcal{I}_\varphi$$

If moreover φ is elliptic i.e. $\varphi(I_{\mathcal{O}_F})$ is finite ↗ supercuspidal packets

then $\mathcal{I}_\varphi = j_! * j^* \mathcal{I}_\varphi$

3) $[L] \in B(G)$ basic $\chi_L: [\text{Sta}(C_F^b) / \mathcal{I}_\varphi(\mathcal{O}_F)] \xrightarrow{\text{given by } \mathcal{E}_L} \text{Bun}$

$$\chi_L^* \mathcal{I}_\varphi = \bigoplus_{\rho \in \widehat{S}_\varphi} \mathcal{I}_{\varphi, \rho}$$

$\rho |Z(\mathbb{Q})^\times| = \chi(L)$

$\mathcal{I}_{\varphi, \rho}$ L-packet for L.L.C. (see Kaletha.)
↗ $\text{Aut}(\text{trivial v.l.}) = G$ as alg. group
↘ $\text{Aut}(\text{---}) = G(\mathcal{O}_F)$ top. group

↖ Big difference with F.G.V.

4) Character sheaf property: $\delta \in G(\mathcal{O}_F) \subset G(L)$, \mathcal{E}_δ defined over \mathbb{F}_q

$$\Rightarrow \chi_\delta^* \mathcal{I}_\varphi \cong \text{Frob.}$$

↖ Weil sheaf structure

Frob acts like δ

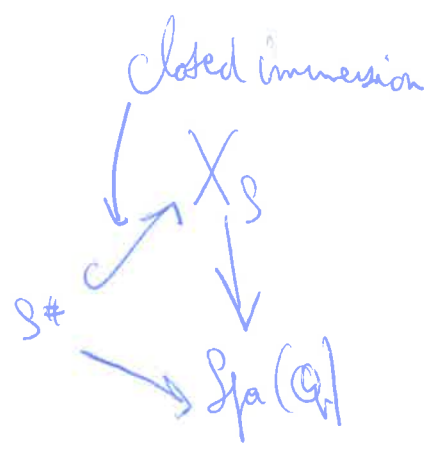
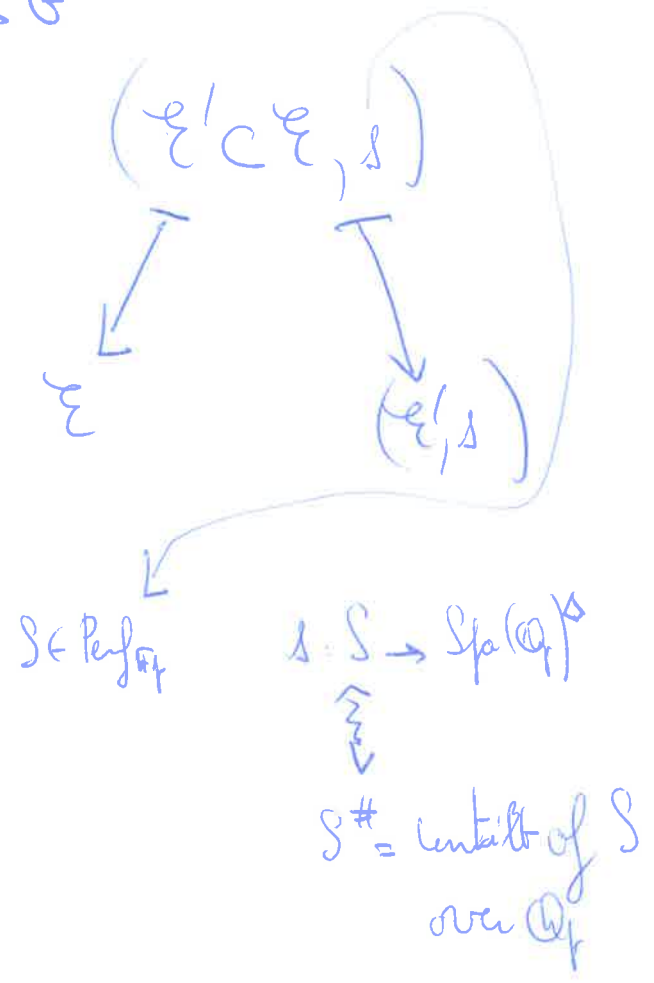
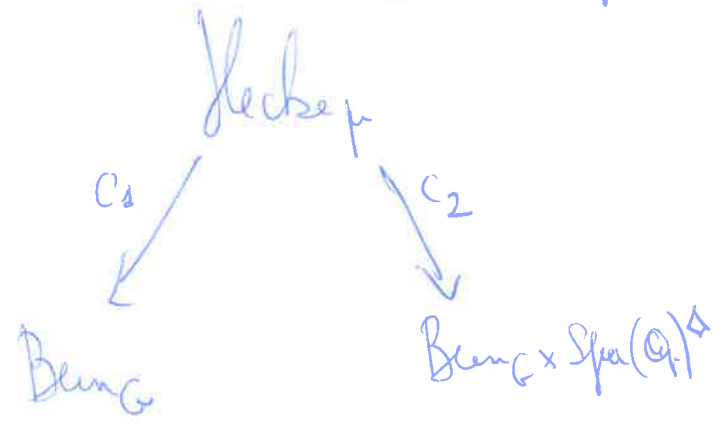
$$G(\mathcal{O}_F) \text{ s.s.-reg} \longrightarrow \overline{\mathbb{Q}_\ell}$$

$$\delta \longmapsto \text{tr}(\text{Frob}; \chi_\delta^* \mathcal{I}_\varphi) = \bigoplus_{\rho \in \widehat{S}_\varphi} \mathcal{I}_{\varphi, \rho}(\delta)$$

is the stable distrib on $G(\mathcal{O}_F)$ associated to φ

↖ real trace if \mathcal{I}_φ unramified mod center.

5) Hecke property $\mu: G_m \rightarrow G$



$\rightarrow \rho_\mu \in \text{Rep}_{\overline{\mathbb{Q}_p}}(\mathbb{Z}_p)$

$c_2! (c_1^* \mathbb{Z}_p \otimes \rho_\mu) = \mathbb{Z}_p \boxtimes \rho_\mu \circ \rho$
 \uparrow
 $\mathbb{Q}_p \langle \rho, \mu \rangle [\langle \rho, \mu \rangle]$ if μ minuscule

A few things:
 * True for tori. It is equivalent to local class field theory
 * Specializing the Hecke property for minuscule μ at the trivial G -torsor (χ_1^*) one finds back Kottwitz Conjecture

describing the contribution of supercuspidal L-factors to the cohomology of RZ. spaces in terms of local Langlands

* If $G = GL_n$ $\mu = (1, 0, \dots, 0)$

Hecke property + existence of \mathcal{I}_G gives back the conjecture by Scholze in February ($\pi_{\text{HKT}}: Sh \rightarrow P^{n-1}$)

* Purpose: Construct \mathcal{I}_G and thus local Langlands using geometric Langlands methods

difficulty: * What is a perverse sheaf on Bun_G ?

→ use uniformization by $G^{B, I, R}$
 ⇒ it suffices to define what is a perverse sheaf on $G^{B, I, R}$

* Construction of Laxson sheaf via $j_!^*$

* Cryptic remark: $b = \begin{pmatrix} 1 & 0 \\ 0 & \varphi^{-1} \end{pmatrix} \rightsquigarrow \mathcal{E}_b = \mathcal{O}_X \oplus \mathcal{O}_X(1)$
 $\text{Act}(\mathcal{E}_b) = \begin{pmatrix} \mathcal{O}_F^* & (B_{\text{class}}^+)^{\varphi^{-1}} \\ 0 & \mathcal{O}_F^* \end{pmatrix}$

* \mathcal{I}_G

Should act trivially when $l \neq p$ but not when $l = p$