Background on the Curve

F perfectoid char. p. Joint work w/t. Fontaine [illegible] Curve X_F

[3 incarnations]

Adic preperfectoid Curve	Schematical Curve	Diamond Curve
$X_F^{ad} = Y_F/\varphi^{\mathbb{Z}}$ $Y_F = \mathrm{Spa}(W(\mathcal{O}_F)) \setminus V(p[\varpi_F])$ = punctured open disk variable p Coeff. in F	$X_F = \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0(X^{ad}, \mathcal{O}(d))\right)$ $\mathcal{O}(d) = Y \underset{\varphi^{\mathbb{Z}}}{\times} \mathbb{A}^1$ $\varphi^{\mathbb{Z}} \curvearrowright \times p^{-d}$ (line bundle) noetherian regular scheme of dim. 1	$X^{\diamond} = \left(\mathrm{Spa}(F) \times \mathrm{Spa}(\mathbb{Q}_p)^{\diamond}\right)/\varphi^{\mathbb{Z}}$ $\varphi = \mathrm{Frob}_F \times \mathrm{Id} = \mathrm{Id} \times \mathrm{Frob}_{\mathbb{Q}_p^{\diamond}}^{-1}$

F alg. closed $k_F \hookrightarrow \mathcal{O}_F$ (Choose a section of the projection to the residue field)

$$\begin{tikzcd} Y_F \arrow[d] \\ \mathrm{Spa}(L) \end{tikzcd}$$

$L = W(k_F)_{\mathbb{Q}} \circlearrowleft \sigma$

$$\begin{tikzcd} \underbrace{\varphi\text{-Mod}_L}_{\text{Iso-crystals}} \arrow[r] & \mathrm{Bun}_{X^{ad}} & \mathrm{Bun}_X \arrow[l, "\overset{GAGA}{\sim}"'] \\ (D,\varphi) \arrow[r, mapsto] & Y \underset{\varphi^{\mathbb{Z}}}{\times} D \arrow[d] & \\ & Y/\varphi^{\mathbb{Z}} & \end{tikzcd}$$

Th (Fontaine-F.). This is essentially surjective.

Classification of G_torsors

G reductive gp. / $\mathbb{Q}_p$

G_bundles$/X$ = { G-torsors on X (locally trivial for the étale top.)
||
Bundle wt. a G-structure (Tannakian sens) }

$b \in G(L) \longmapsto \mathcal{E}_b$ G-torsor on X_F

preceding functor

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_p} G \arrow[r] & \varphi\text{-}\mathrm{Mod}_L \arrow[r] & \mathrm{Bun}_X \\
(V,\rho) \arrow[r, mapsto] & (V_L, \rho(b)\sigma) &
\end{tikzcd}$$

Th: * Any G_torsor on X is of this form

Kottwitz set

* canonical bijection $\boxed{H^1(X,G) \xrightarrow{\sim} B(G)} = G(L)/G\text{-conjugacy}$

doesnt depend on $b_F \hookrightarrow G_F$

Nice features: * $H^1(\mathbb{Q}_p, G) \subset B(G)$

$\qquad ||$

$\{[b] \in B(G) \mid \nu_b = 1\}$ = unit root G_isocrystals

$\uparrow$ slope morphism (given Dieudonné_Manin slope decomposition)

given by $H^1(\mathbb{Q}_p, G) \subset H^1(X, G)$

$\uparrow$ pullback of G_torsors

* G quasi_split. b basic $\Longleftrightarrow \mathcal{E}_b$ semi_stable in Atiyah-Bott sens

There's a dictionary between Kottwitz and Atiyah-Bott reduction

Bun_G

Perf = $\mathbb{F}_p$-perfectoid spaces + pro-étale topology (or faithfull)

Ψ

$S \rightsquigarrow X_S^{ad}$ = adic space$/\mathbb{Q}_p$

= family of curves of the preceding type parametrized by S

Th: $S \mapsto \{\underbrace{G\text{-bundles on } X_S}_{\mathrm{Rep}_{\mathbb{Q}_p} G \xrightarrow{\otimes\text{-exact}} v.b. \text{ on } X_S}\}$ is a stack (Scholze: even for the faithfull top.)

Let's try to understand this stack

$$\underbrace{|\mathrm{Bun}_G \times \overline{\mathbb{F}}_p|}_{\text{top. space open} = |\mathcal{U}|,\ \mathcal{U} = \text{open substack}} \xrightarrow{\sim} B(G)$$

↑ preceding classification

Connected components: $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma = X^*(Z(\hat{G})^\Gamma)$

degree of a v.b. if $G = GL_n$

↑ can be interpreted as a generalized first Chern class of a v.b.

Th (in progress). This is locally constant on $|\mathrm{Bun}_G \times \overline{\mathbb{F}}_p|$

Conjecture: Fibers of K = Connected Components

(should not be difficult)

H.N. stratification :

$$|\mathrm{Bun}_G \times \overline{\mathbb{F}}_p| \longrightarrow \mathrm{Hom}(\mathbb{D}_{\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) / G(\overline{\mathbb{Q}}_p)\text{-conj.}$$

$$B(G) \ni [b] \longmapsto [\nu_b]$$

Kedlaya-Liu (GL_n): this is semi-continuous

s.s. locus = open strata.

S.S. locus: $[b] \in B(G) \rightsquigarrow \mathcal{E}_b = G$-torsor on $X_{\mathbb{C}_p^b}$

$$\underset{\substack{\shortparallel \\ \text{diamond}}}{\widetilde{J}_b} := \mathrm{Aut}(\mathcal{E}_b) = \text{group sheaf on } \mathrm{Perf}_{\mathbb{C}_p^b}$$

$$\pi_0(\widetilde{J}_b) = J_b(\mathbb{Q}_p) = \sigma\text{-Centralizer of } b \text{ in } G(L)$$

$$\widetilde{J}_b^\circ = \text{unipotent} \quad (\text{group-diamond})$$

Ex. $\mathrm{Aut}(\mathcal{O}_X \oplus \mathcal{O}_X(1)) = \begin{pmatrix} \mathbb{Q}_p^\times & (B_{cris}^+)^{\varphi = p} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ $\leftarrow$ cryptic remark

$\mathcal{D}(\mathbb{Q}_p)$ (loc. analytic distributions)

b basic $\Rightarrow \widetilde{J}_b^\circ = \{1\}$

$\mathrm{Aut}(\mathcal{E}_b) = \underline{J_b(\mathbb{Q}_p)}$] any loc. profinite top. group defines a group sheaf on the pro-étale site

$\uparrow$

inner form of G.

Using Kedlaya-Liu : $\underline{\text{Th}}$: $[b]$ basic $\Rightarrow$ $\mathrm{Bun}_{G/\overline{\mathbb{F}}_q}^{[b]} = B(J_b(\mathbb{Q}_p))$

↑ GL_n, $b=1$

s.s. strata = zero dimensional

dim. stratum associated to $\mathcal{O}_X \oplus \mathcal{O}_X(1) = -1$

deeper in H.N. strata $\Rightarrow$ the dimension goes to $-\infty$.

Uniformization à la Beauville-Laszlo:

$$\mathrm{Gr}^{B_{dR}} \longrightarrow \mathrm{Bun}_G \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

↑ going back to the schematical curve in a relative situation (Kedlaya-Liu)

$\underline{\text{Th}}$: F alg. closed. $\infty \in |\underbrace{X_F}_{\text{schematical curve}}|$

$\mathcal{E}$ = G-torsor on X_F. Then $\mathcal{E}_{|X_F \setminus \{\infty\}}$ is trivial.

↳ analog of Drinfeld-Simpson

Should imply, using $\underbrace{\mathrm{Gr}^{B_{dR}} \text{ is a diamond}}_{\text{Scholze.}}$, Bun_G = perfectoid stack (has a pro-étale cover that is a presentation by a perfectoid space)

The Conjecture

Motivation: * F perfectoid field

$$\mathrm{Pic}^0(X_F) = \mathrm{Hom}\left(\pi_1^{geo}(X_F), \mathbb{Q}_p^\times\right)$$

⇝ starting point of geometric class field theory of X = "classical curve"

* Kottwitz conjecture (generalizing Vogan's conjecture → pure inner forms): local Langlands correspondence for $\underbrace{\text{extended pure inner forms of } G}_{J_b \text{ with } b \text{ basic}}$

→ Kaletha work (depth 0 case)

Rem: b basic J_b = inner form of G (not pure in general)

$J_b \times X$ = pure inner form of $G \times X$

twist by $[\mathcal{E}_b] \in H^1(X, G)$

* "$\underbrace{\mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}}}_{\pi_1 = \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)} = X_{\mathbb{F}_1}$" ⇝ "$X_S = S \times \mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}}$"

* Frenkel - Gaitsgory - Vilonen ...

$\widehat{G}$ = $\overline{\mathbb{Q}}_\ell$-Langlands dual ⇝ group in $(\mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}})_{et}$

$\circlearrowleft$ $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$

Canonically defined via geometric Satake (Scholze via $\mathrm{Gr}^{B_{dR}}$)

Conjecture (rough form):

Groupoid of $\hat{G}$-torsors on $\mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}}$ $\longrightarrow$ Perverse sheaves on $Bun_{\mathbb{Q}_p^\diamond/\varphi^{\mathbb{Z}}}$ - Hecke equivariants

Conjecture: G quasi-split - (B,ψ) = Whittaker datum

$$\mathcal{L}_G = \left[\ \mathrm{Hom}\left(W_{\mathbb{Q}_p}, {}^L G\right)/\hat{G}\text{-Conj.}\right]$$ = groupoid of Langlands parameters

$\mathcal{L}_G^{disc}$ = discrete ~~Langlands~~ parameters $\longrightarrow$ $S_\varphi/Z(\hat{G})^\Gamma$ finite

$\varphi : W_{\mathbb{Q}_p} \to {}^L G$ $\qquad$ $S_\varphi = \mathrm{Cent}_{\hat{G}}(\varphi) \supset Z(\hat{G})^\Gamma$

There exist a functor

$$\begin{tikzcd}
\mathcal{L}_G^{disc} \arrow[r] & \text{Weil-Perv}\left(Bun_{G/\bar{\mathbb{F}}_p}\right) \\
\varphi \arrow[r, mapsto] & \mathcal{F}_\varphi \\
S_\varphi \arrow[u, "\circlearrowleft"] \arrow[r] \arrow[loop left] & S_\varphi \arrow[u, "\circlearrowleft"]
\end{tikzcd}$$

Such that:

1) If $\xi \in X^*(Z(\hat{G})^\Gamma)$ and $Bun^{\xi}_{G/\bar{\mathbb{F}}_p} = \kappa^{-1}(\xi)$ (associated connected component)

$\mathcal{F}_\varphi|_{Bun^\xi}$

$\circlearrowleft$

$S_\varphi \supset Z(\hat{G})^\Gamma$ acts via ξ

2) $\mathcal{F}_{\varphi|\mathrm{Bun}^{ss}}$ is a local system and if $j : \mathrm{Bun}^{ss} \overset{\text{open}}{\hookrightarrow} \mathrm{Bun}$

$$\mathcal{F}_\varphi = j_{!*} j^* \mathcal{F}_\varphi$$

supercuspidal packets

If moreover φ is elliptic i.e. $\varphi(I_{\mathbb{Q}_p})$ is finite

then $\mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi$

3) $[b] \in B(G)$ basic

given by $\mathcal{E}_b$

$$x_b : \left[\mathrm{Spa}(\mathbb{C}_p^b) / J_b(\mathbb{Q}_p)\right] \longrightarrow \mathrm{Bun}$$

$$x_b^* \mathcal{F}_\varphi = \bigoplus_{\substack{\rho \in \widehat{S_\varphi} \\ \rho|_{Z(\hat{G})^\Gamma} = \kappa(b)}} \mathcal{F}_{\varphi,\rho}$$

$\circlearrowleft$

$J_b(\mathbb{Q}_p) \times S_\varphi$

$\{\mathcal{F}_{\varphi,\rho}\}_\rho$ L-packet for L.L.C. (see Kaletha)

Big difference with F.G.V. → $\mathrm{Aut}(\text{trivial v.b.}) = G$ as alg. group
→ $\mathrm{Aut}(\text{—}) = G(\mathbb{Q}_p)$ top. group.

4) Character sheaf property: $\delta \in G(\mathbb{Q}_p) \subset G(L)$, $\mathcal{E}_\delta$ defined over $\mathbb{F}_p$

$$\underset{\text{Weil sheaf structure}}{\Longrightarrow} \quad x_\delta^* \mathcal{F}_\varphi \circlearrowleft \mathrm{Frob}.$$

Frob acts like δ

real trace if $\overline{J_\delta}$ anisotropic mod center.

$$G(\mathbb{Q}_p)_{s.s.\text{-}reg} \longrightarrow \overline{\mathbb{Q}}_\ell$$

$$\delta \longmapsto \mathrm{tr}(\mathrm{Frob}; x_\delta^* \mathcal{F}_\varphi) = \Theta_{x_\delta^* \mathcal{F}_\varphi}(\delta)$$

is the stable distrib on $G(\mathbb{Q}_p)$ associated to φ

5) Hecke property

$\mu : \mathbb{G}_m \to G$

$$\begin{tikzcd} & \mathrm{Hecke}_\mu \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond \end{tikzcd}$$

$$\begin{tikzcd} & (\mathcal{E}' \subset \mathcal{E}, s) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mathcal{E} & & (\mathcal{E}', s) \end{tikzcd}$$

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\qquad s : S \to \mathrm{Spa}(\mathbb{Q}_p)^\diamond$

$\Updownarrow$

$S^\# =$ untilt of S over $\mathbb{Q}_p$

closed immersion

$$\begin{tikzcd} & X_S \arrow[d] \\ S^\# \arrow[ur, hook] \arrow[r] & \mathrm{Spa}(\mathbb{Q}_p) \end{tikzcd}$$

$$c_{2!}\big(c_1^* \mathcal{F}_\varphi \otimes \underset{\overline{\mathbb{Q}}_\ell(\langle\rho,\mu\rangle)[\langle 2\rho,\mu\rangle] \text{ if } \mu \text{ minuscule}}{\underset{\uparrow}{K_\mu}}\big) = \mathcal{F}_\varphi \boxtimes \overset{r_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)}{\overset{\nearrow}{r_\mu}} \circ \varphi$$

A few things: * True for tori. It is equivalent to local class field theory

* Specializing the Hecke property for minuscule μ at the trivial G-torsor (x_1^*) one finds back Kottwitz Conjecture

describing the contribution of supercuspidal L-packets to the cohomology of R.Z. spaces in terms of local Langlands

* If $G = GL_n$ $\mu = (1, 0, \dots, 0)$
Hecke property + existence of $\mathcal{F}_\varphi$ gives back the conjecture by Scholze in February ($\pi_{HT}: Sh \to \mathbb{P}^{n-1}$)

* Purpose: Construct $\varphi \mapsto \mathcal{F}_\varphi$ and thus local Langlands using geometric Langlands methods

<u>difficulty</u>: * What is a perverse sheaf on Bun_G?
→ use uniformization by $G^{B_{dR}}$
⇒ it suffices to define what is a perverse sheaf on $G^{B_{dR}}$
* Construction of Laumon sheaf via $j_{!*}$

* Cryptic remark: $b = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \rightsquigarrow \mathcal{E}_b = \mathcal{O}_X \oplus \mathcal{O}_X(1)$

$Aut(\mathcal{E}_b) = \begin{pmatrix} \mathbb{Q}_p^\times & (B^+_{cris})^{\varphi = p} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$

$\circlearrowleft$
$i_b^* \mathcal{F}_\varphi$

should act trivially
when $l \neq p$ but not
when $l =$